angle-of-attack, respectively and they can be obtained from tables in Ref. 10 for various NACA airfoils

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Flight Path Optimization with Multiple Time Scales

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The use of multiple time scales[1] in the optimization of aircraft flight paths is examined via asymptotic expansion in several parameters.[2,3] It turns out that, with more than two scales, the boundary layers develop their own boundary layers and the problem decouples into several problems of lower order, some furnishing performance indices for others. In the following example, optimal aircraft motion on three time scales is examined in cascaded boundary-layer approximation.

Introduction

THE idea of time-scale separation in vehicle dynamics is expounded in an excellent paper by Ashley[1] based upon asymptotic expansion techniques from fluid dynamics research, as reported in the references cited. The same intuitive insight regarding multiple time scales underlies the present treatment of optimal aircraft flight; however, the asymptotic methods of ordinary differential equations[2,3] are adopted. There appears to be a close conceptual kinship between the two bodies of literature but only limited cross-referencing. An earlier publication in the present vein has examined the aircraft "energy climb" and a related turning-flight approximation in terms of motion on two time scales[4]; another one has explored the adaptation of asymptotic expansion theory of ordinary differential equations to a simple variational two-point boundary-value problem.[5] The main point of the present paper is the decoupling of a high-order three-dimensional aircraft flight problem into several (specifically three) lower-order problems, with the possibility of further extension to rigid-body and control-motion problems which take place on additional, faster, time scales.

Asymptotic Expansion Formulation

The equations of motion for three-dimensional aircraft flight are

$$\epsilon_2 \dot{h} = V \sin\gamma \tag{1}$$

$$\epsilon_2 \dot{\gamma} = (g/V)[(L + \epsilon_2 T \sin\alpha)\cos\mu/(W_0 + \epsilon_2 \Delta W) - \cos\gamma] \tag{2}$$

$$\epsilon_1 \dot{E} = \{[(T - D)V + \epsilon_2 TV(\cos\alpha - 1)]/(W_0 + \epsilon_2 \Delta W)\} \tag{3}$$

$$\epsilon_1 \dot{\chi} = gL \sin\mu/V(W_0 + \epsilon_2 \Delta W)\cos\gamma \tag{4}$$

$$\dot{x} = V \cos\gamma \sin\chi \tag{5}$$

$$\dot{y} = V \cos\gamma \cos\chi \tag{6}$$

$$\Delta\dot{W} = -Q(V,h) \tag{7}$$

These apply for full-throttle, zero side-force flight over a flat Earth. The right members incorporate, for $\epsilon_2 = 0$, the assumptions of constant weight and thrust directed along the path.[4] $E \equiv h + V^2/2g$ is specific energy, χ heading angle, γ path angle to horizontal, and μ bank angle. The symbol V for velocity should be regarded as merely convenient shorthand for $V \equiv [2g(E - h)]^{1/2}$.

An asymptotic expansion in the parameters ϵ_1 and ϵ_2, which fall in the ranges $0 \leq \epsilon_1 \leq 1$, $0 \leq \epsilon_2 \leq 1$, is contemplated. Following Tihonov,[2,3] it is supposed that the ratio $\epsilon_2/\epsilon_1 \to 0$ as $\epsilon_1 \to 0$. The parameters ϵ_1 and ϵ_2 are normalized analogues of the small parameters of fluid mechanics boundary-layer theory, e.g., in a flow problem ϵ might be defined as $\epsilon \equiv \nu/\bar{\nu}$, where ν is the viscosity and $\bar{\nu}$ its numerical value for a particular problem of interest. The hope of an expansion procedure is that the solutions sought for $\epsilon_i = 1$ differ little from those obtained for $\epsilon_i = 0$ except in short-time intervals of transition, "boundary layers" in an extended sense of the term.

The dependence of the *left* members of the system of state Eqs. (1–7) upon ϵ_1 and ϵ_2 is of primary importance in furnishing reductions in order. This order reduction with the vanishing ϵ_i is the defining feature of *singular perturbation problems.*

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The ϵ dependence of the *right* members represents little more than a convenience; it imbeds some approximations in the reduced-order system, in the manner of conventional perturbation theory.

The Euler equations are

$$\epsilon_2 \dot{\lambda}_h = -\partial H/\partial h \tag{8}$$

$$\epsilon_2 \dot{\lambda}_\gamma = -\partial H/\partial \gamma \tag{9}$$

$$\epsilon_1 \dot{\lambda}_E = -\partial H/\partial E \tag{10}$$

$$\epsilon_1 \dot{\lambda}_\chi = -\partial H/\partial \chi \tag{11}$$

$$\dot{\lambda}_x = -\partial H/\partial x \tag{12}$$

$$\dot{\lambda}_y = -\partial H/\partial y \tag{13}$$

$$\dot{\lambda}_W = -\partial H/\partial W \tag{14}$$

$$\partial H/\partial \alpha = 0 \tag{15}$$

where

$$H \equiv \sum_{j=1}^{n} \lambda_j f_j \tag{16}$$

the f_j being the right members of the state equations (1–7).

Time Scale Separation

The parameters ϵ_1 and ϵ_2 are introduced to separate motions taking place on time scales essentially different from one another. With $\epsilon_2 = 0$, the relatively fast kinetic-potential energy interchange is discarded in favor of a model featuring instantaneous interchange. When, in addition, $\epsilon_1 = 0$, then even the slower processes of heading and total-energy change become instantaneous, yielding a highly-simplified rectilinear motion model. To satisfy the requirements of the Tihonov theory that $\epsilon_2/\epsilon_1 \to 0$ as $\epsilon_1 \to 0$, a single parameter ϵ could be employed and ϵ_1 taken as ϵ, ϵ_2 as ϵ^2. If rigid-body short-period motion were to be included in the model, a third parameter $\epsilon_3 = \epsilon^3$ would be introduced.

Procedures for scale separation in general problems have not been established, and the following short discussion will necessarily be somewhat sketchy and speculative. Historically, the parameter ϵ was introduced as a multiplicative factor upon the highest-order term of a differential equation. In the present setting, ϵ is an interpolation parameter introduced artificially into a system of first-order differential equations. Sorting out fast and slow motions by judicious choice of variables should be considered simultaneously with the task of introducing ϵ factors, e.g., the adoption of energy as a state variable, which is more slowly varying than either of its ingredients, V and h, in the aircraft motion equations.

With linear constant-coefficient systems, the groupings of eigenvalues of the characteristic polynomial will aid useful distinction between fast and slow. Reduction of coupling between differential equations of fast and slow variables is a desirable goal in selection of variables, in that eigenvalues of the reduced and boundary-layer systems will then not differ much from those of the original system, and the possibility of a good approximation by a composite comprised of only low-order terms in ϵ is enhanced. Means of systematically choosing variables in both linear and nonlinear systems are of research interest.

In high-order systems, such as the aircraft model at hand, separation into more than two groups by the introduction of additional parameters may be appropriate. The grouping adopted in the preceding equations seems fairly natural with, perhaps, a slight arbitrariness in the common time-scale choice for heading and total-energy changes. It might be appropriate for some situations, and/or as a possible gross simplification, to idealize total-energy changes as fast compared to heading changes, thus effecting still further mathematical separation; this is pursued in the study of Ref. 6.

Reduced System: Optimal Steady Flight

Consider first the problem of minimum-time motion between end states fixed except for fuel expenditure open. One begins by solving the optimization problem for the reduced system, $\epsilon_1 = \epsilon_2 = 0$ (i.e., the rectilinear motion model), then performs corrections for the initial and terminal transients in boundary-layer approximation. One obtains

$$\sin\bar{\gamma} = 0, \quad \cos\bar{\gamma} = 1 \tag{17}$$

$$L = W_0 \tag{18}$$

$$\bar{T} = D \tag{19}$$

$$\sin\bar{\mu} = 0, \quad \cos\bar{\mu} = 1 \tag{20}$$

$$\bar{x} = x_0 + t\bar{V}\sin\bar{\chi} \tag{21}$$

$$\bar{y} = y_0 + t\bar{V}\cos\bar{\chi} \tag{22}$$

Here $\bar{T}$ is full-throttle thrust; superscribed bars generally denote the solution of the reduced state-Euler system. The multipliers $\bar{\lambda}_x$ and $\bar{\lambda}_y$ are constant and $\bar{\lambda}_W = 0$. $\bar{E}$, $\bar{h}$, and $\bar{\chi}$ minimize H subject to Eqs. (17–20). This is an exercise in determining the steady-state level-flight performance envelope and in selecting the maximum level-velocity point.[7] The multipliers $\bar{\lambda}_x$ and $\bar{\lambda}_y$ are chosen so that $\bar{\chi}$ is the proper constant heading to reach the specified terminal point; they are scaled in magnitude according to the transversality condition $\bar{H} = 1$.

Other problems such as minimum fuel, maximum range, etc., exhibiting rectilinear motion over a central portion of the path, can be treated similarly. Attention is now directed to the transitions at either end.

Boundary-Layer System: Energy-Heading Transient

Consider transition to and from the rectilinear motion in boundary-layer approximation. Near the initial point, a stretched time scale is introduced via the adoption of a time variable $\tau = t/\epsilon_1$.[2,3] For small τ, one obtains a reduction in the order of the state-Euler system for $\epsilon_1 = 0$ by virtue of $dx/d\tau = dy/d\tau = dW/d\tau = d\lambda_x/d\tau = d\lambda_y/d\tau = d\lambda_W/d\tau = 0$. One is thus led to an energy model for optimal transitional flight similar, except for performance index, to that of Ref. 4:

$$V\sin\gamma = 0 \tag{23}$$

$$(g/V)[L\cos\mu/W_0 - \cos\gamma] = 0 \tag{24}$$

$$dE/d\tau = V(T - D^*)/W_0 \tag{25}$$

$$d\chi/d\tau = gL\sin\mu/VW_0 \tag{26}$$

$$\frac{d}{d\tau}\lambda_E = -\frac{\partial}{\partial E}\left[\lambda_E \frac{V(T-D^*)}{W_0} + \lambda_\chi \frac{gL^*\sin\mu}{VW_0}\right] - \frac{\partial}{\partial E}\{\bar{\lambda}_x V\sin\chi + \bar{\lambda}_y V\cos\chi - \bar{\lambda}_W Q\} \tag{27}$$

$$(d/d\tau)\lambda_\chi = -(\partial/\partial\chi)\{\bar{\lambda}_x V\sin\chi + \bar{\lambda}_y V\cos\chi\} \tag{28}$$

$$\partial H^*/\partial\mu = 0 \tag{29}$$

$$\partial H^*/\partial h = 0 \tag{30}$$

where

$$H^* \equiv \lambda_E(T - D^*)V/W_0 + \lambda_\chi gL^*\sin\mu/VW_0 + \{\bar{\lambda}_x V\sin\chi + \bar{\lambda}_y V\cos\chi - \bar{\lambda}_W Q\} \tag{31}$$

For simplicity, it has been elected to account for Eqs. (16)

and (17) in the remaining equations by substitution of $\gamma = 0$ and evaluating drag D for $L = L^* = W_0/\cos\mu$, designating it D^*, rather than to employ multipliers λ_h and λ_γ.

The multipliers $\bar{\lambda}_x$, $\bar{\lambda}_y$, $\bar{\lambda}_W$ are known constants from the optimal steady flight solution of the reduced system. When they are not constant, as would be the case, for example, if winds aloft were included in the reduced-system model, the initial values would appear in the boundary-layer state-Euler system. These are, nonetheless, merely known constants for purposes of the boundary-layer calculation.

The terms set off in brackets, $\{\cdot\}$, for ready identification, correspond to a performance index

$$\int_0^\infty \{\bar{\lambda}_x V \sin\chi + \bar{\lambda}_y V \cos\chi - \bar{\lambda}_W Q\} d\tau \tag{32}$$

The reduced system furnishes the form of this index, and its numerical solution the required initial values of the barred quantities appearing in it. Variables satisfying the boundary-layer equations will be denoted by superscript*.

Initial values of multipliers not determined by transversality conditions must be chosen to suppress unstable boundary-layer transients. When this is not possible, the boundary-layer approximation approach fails.[5] The determination of conditions under which the possibility can be guaranteed a priori is an open research question. The limited evidence from study of the low-order system of Ref. 5, however, is encouraging in that a condition found as necessary—strengthened Legendre-Clebsch at the endpoints—is not unduly restrictive.

Sublayer System: Energy Interchange

Because the left members of Eqs. (1, 2, 8, and 9) never entered the generation of the system Eqs. (23–31), there is a mismatch of that system's solution with specified initial values. Transitions in sub-boundary-layer approximation are required. These are described by a further stretching of scale, $T = \tau/\epsilon_2$, which leads, for $\epsilon_2 = 0$, to $dE/dT = d\chi/dT = d\lambda_E/dT = d\lambda_\chi/dT = 0$, and to the system

$$dh/dT = V^* \sin\gamma \tag{33}$$

$$d\gamma/dT = (g/V^*)[(L\cos\mu/W_0) - \cos\gamma] \tag{34}$$

$$\frac{d\lambda_h}{dT} = -\frac{\partial \hat{H}}{\partial h} = -\frac{\partial}{\partial h}\left[\lambda_h V^* \sin\gamma + \lambda_\gamma \frac{g}{V^*}\left(\frac{L\cos\mu}{W_0} - \cos\gamma\right)\right] - \frac{\partial}{\partial h}\left\{\lambda_{E_0}{}^* \frac{V^*(T-D)}{W_0} + \lambda_{\chi_0}{}^* \frac{gL\sin\mu}{V^* W_0 \cos\gamma}\right\} \tag{35}$$

$$\frac{d\lambda_\gamma}{dT} = -\frac{\partial \hat{H}}{\partial \gamma} = -\frac{\partial}{\partial \gamma}\left[\lambda_h V^* \sin\gamma + \lambda_\gamma \frac{g}{V^*}\left(\frac{L\cos\mu}{W_0} - \cos\gamma\right)\right] - \frac{\partial}{\partial \gamma}\left\{\lambda_{E_0}{}^* \frac{V^*(T-D)}{W_0} + \lambda_{\chi_0}{}^* \frac{gL\sin\mu}{V^* W_0 \cos\gamma}\right\} \tag{36}$$

$$\partial\hat{H}/\partial\alpha = 0 \tag{37}$$

$$\partial\hat{H}/\partial\mu = 0 \tag{38}$$

Here

$$V^* \equiv [2g(E_0{}^* - h)]^{1/2} \tag{39}$$

and

$$\hat{H} \equiv \left[\lambda_h V^* \sin\gamma + \lambda_\gamma \frac{g}{V^*}\left(\frac{L\cos\mu}{W_0} - \cos\gamma\right)\right] + \left\{\lambda_{E_0}{}^* \frac{V^*(T-D)}{W_0} + \lambda_{\chi_0}{}^* \frac{gL\sin\mu}{V^* W_0 \cos\gamma}\right\} \tag{40}$$

The terms set off in brackets, $\{\cdot\}$, again for ready identification, correspond to a performance index

$$\int_0^\infty \left\{\lambda_{E_0}{}^* \frac{V^*(T-D)}{W_0} + \lambda_{\chi_0}{}^* \frac{gL\sin\mu}{V^* W_0 \cos\gamma}\right\} dT \tag{41}$$

for the sublayer optimization, the quantities subscripted zero being the initial values of the boundary-layer solution.

Composition of the Solution

Evidently there will be, at the terminus of the rectilinear motion, a terminal layer and a sublayer. These are treated similarly in a reversed-time framework. Exceptionally, sublayers may also attach themselves to "corners" in the interior of reduced-order solutions, as in transition between subsonic and supersonic excess-power peaks in "energy climbs."[4]

Concluding Remarks

There is a rigorous theory of expansions for initial-value problems, including a foundation based largely upon stability hypotheses (Sec. 39 of Ref. 3) and a procedure for higher-order corrections (Sec. 40 of Ref. 3), which applies to the two-time-scale situation. The multiple time-scale (multi-layer) case (Ref. 2 and, briefly, p. 258 of Ref. 3) has not been extensively treated; however, the present attempt to describe how a treatment might proceed for the optimal aircraft flight application is intended to focus attention on the possibilities of multilayer approximation, which are seemingly endless. One could, for example, continue cascading boundary layers in the airplane maneuvering problem to derive performance indices for the short-period rigid-body motions, and even for the control-surface actuator responses!

Two research areas of particular interest for future utilization of multilayer approximation are the two-point boundary-value aspects, and corresponding restrictions on applicability, and the development of computational methods tailored to the scheme for decoupling into lower-order problems.

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